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Matrix-Theoretic Criteria for the Quasi-Convexity and Pseudo-Convexity of Quadratic Functions*

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1. INTRODUCTION

It is well known that quasi-convexity and pseudo-convexity play a "natural" role in nonlinear programming theory. (See [6, Chapters 9 and 10].) Nevertheless, it is said that these notions lack utility because they have defining conditions involving infinitely many inequalities and are not easily checked. Our aim in this paper is to prove that testing the quasi-convexity (pseudo-convexity) of a quadratic function on the non-negative (semipositive) orthant can be reduced to an examination of finitely many conditions. To accomplish this we borrow heavily from two recent works of Martos [7, 8]. In some respects, our effort parallels that of Cottle *et al.* [3] on copositive quadratic forms.

In Section 2 we briefly review the general definitions of quasi-convexity and pseudo-convexity. Next we specialize to the case of quadratic forms and state the definitions of positive subdefiniteness and strict positive subdefiniteness introduced by Martos [7]. The theorems on the subject obtained by Martos are summarized in Section 3. Our main results on quadratic forms are developed in Section 4, and in Section 5 we turn to the question of quadratic functions. Some brief remarks on nonconvex quadratic programming are presented in Section 6.

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2. REVIEW OF DEFINITIONS

A real-valued function ϕ defined on a convex set S is *quasi-convex* on S if and only if

$$\{x \in S | \phi(x) \leqq \xi\} \quad \text{is convex for all} \quad \xi \in E^1.$$

This property is partially responsible for the theoretical usefulness of quasi-convex functions as constraints in a nonlinear program. For differentiable functions a "gradient inequality" serves to define quasi-convexity. A differentiable function ϕ is *quasi-convex* on the convex set S if and only if, for all $x, y \in S$,

$$\phi(y) \leqq \phi(x) \quad \text{implies} \quad \nabla\phi(x)(y - x) \leqq 0. \tag{2.1}$$

The concept of quasi-convexity was introduced by Nikaidô [9] and subsequently developed by Arrow and Enthoven [1].

A real-valued differentiable function ϕ on a convex set S is *pseudo-convex* if and only if the gradient inequality

$$\nabla\phi(x)(y - x) \geqq 0 \quad \text{implies} \quad \phi(y) \geqq \phi(x) \tag{2.2}$$

for all $x, y \in S$. The significance of (2.2) for nonlinear programming is that local minima of ϕ on S are global minima if ϕ is pseudo-convex. This concept was first introduced by Tuy [11] and later more fully by Mangasarian [5, 6]. See also Ponstein [10].

A differentiable convex function is pseudo-convex and a pseudo-convex function is quasi-convex. The implied set-theoretic inclusions are proper, although for certain types of functions over certain sets the distinction does not exist. We shall return to this point in Section 5.

Throughout this paper, D will stand for a real symmetric matrix of order n. Associated with D is the *quadratic form*

$$\psi(x) = x^T D x$$

defined for all $x \in E^n$.

The necessary and sufficient conditions for ψ to be *quasi-convex* on a convex set S are

$$y^T D y \leqq x^T D x \quad \text{implies} \quad x^T D(y - x) \leqq 0 \tag{2.3}$$

for all $x, y \in S$. This application of (2.1) to the function ψ suggests that *a priori* there could be infinitely many inequalities to check.

Likewise, the necessary and sufficient conditions for ψ to be pseudo-convex on S are

$$x^T D(y - x) \geqq 0 \quad \text{implies} \quad y^T D y \geqq x^T D x \tag{2.4}$$

for all $x, y \in S$; these, too, could represent an infinity of conditions.

We hasten to point out that the convexity of ψ on S is equivalent to $(y - x)^T D(y - x) \geqq 0$ for all $x, y \in S$ and, when the carrying plane of S is known, the test is finite. See [2]. Moreover, the property known as *copositivity*, i.e.,

$$x^T D x \geqq 0 \quad \text{for all} \quad x \geqq 0$$

is finitely testable even though stated by an infinite number of inequalities. See [3].

Recall that a nonzero, nonnegative vector x is called *semipositive*; we denote this by writing $x \geqslant 0$ (rather than $x \geqq 0$ which means nonnegative and possibly 0). Naturally x is seminegative ($x \leqslant 0$) if and only if $-x \geqslant 0$. The same kind of terminology applies to real matrices. For example, $D \leqslant 0$ means that D is nonpositive (entry by entry) but not the zero matrix.

In [7], Martos has identified a class of matrices D and corresponding quadratic forms $\psi(x) = x^T D x$ called *positive subdefinite*. Their defining property is

$$x^T D x < 0 \quad \text{implies} \quad Dx \geqslant 0 \quad \text{or} \quad Dx \leqslant 0. \tag{2.5}$$

Moreover, the quadratic form ψ is *strictly positive subdefinite* if and only if

$$x^T D x < 0 \quad \text{implies} \quad Dx > 0 \quad \text{or} \quad Dx < 0. \tag{2.6}$$

It is evident that positive semidefinite quadratic forms are strictly positive subdefinite (by default), and strictly positive subdefinite quadratic forms are positive subdefinite. Thus, in order to exclude the positive semidefinite quadratic forms, Martos inserts the word "merely" before "positive subdefinite."

The modifications of (2.1)–(2.6) required for quasi-concavity, pseudo-concavity, and (strict) negative subdefiniteness are obvious and are therefore omitted.

3. THE THEOREMS OF MARTOS

Our purpose in this section is simply to summarize the principal results accumulated by Martos in [7] and [8]. We shall omit several lemmas and all the proofs.

The first of these results is a set of necessary conditions.

THEOREM 3.1. [Martos, 7, Theorem 1]. *If $\psi(x) = x^T Dx$ is merely positive subdefinite, then*

(i) $D \leqslant 0$,

(ii) *D has exactly one (simple) negative eigenvalue, and*

(iii) *the corresponding eigenvector is either semipositive or seminegative.*

In the next section, we shall furnish a converse to this theorem. The second result is

THEOREM 3.2 [Martos, 7, Theorem 2]. *A merely positive subdefinite quadratic form $\psi(x) = x^T Dx$ is strictly merely positive subdefinite if and only if D does not contain a row (or column) of zeros.*

Martos' paper [7] concludes with two theorems that unify the concepts defined in Section 2 above.

THEOREM 3.3 [Martos, 7, Theorem 4]. *The quadratic form $\psi(x) = x^T Dx$ is quasi-convex on the nonnegative orthant, E_+^n, if and only if it is positive subdefinite.*

THEOREM 3.4 [Martos, 7, Theorem 5]. *The quadratic form $\psi(x) = x^T Dx$ is pseudo-convex on the semipositive orthant, $E_+^n \backslash 0$, if and only if it is strictly positive subdefinite.*

In the other paper [8], Martos deals with quadratic functions and programs. He first obtains the following fact.

THEOREM 3.5 [Martos, 8, Theorem 1]. *The quadratic function $\phi(x) = \frac{1}{2}x^T Dx + c^T x$ is quasi-convex on the nonnegative orthant if and only if, for all $v \in E^n$,*

$$v^T Dv < 0 \quad \textit{implies} \quad \begin{bmatrix} Dv \\ c^T v \end{bmatrix} \geqslant 0 \quad \textit{or} \quad \begin{bmatrix} Dv \\ c^T v \end{bmatrix} \leqslant 0. \tag{3.1}$$

From the standpoint of quadratic programming it is important to characterize the quadratic functions $\phi(x) = \frac{1}{2}x^T Dx + c^T x$ that are quasi-convex on the nonnegative orthant. Hence the following is of interest.

THEOREM 3.6 [Martos, 8, Theorem 2]. *If $\phi(x) = \frac{1}{2}x^TDx + c^Tx$ is quasi-convex but not convex on the nonnegative orthant, then*

(i) $D \leqslant 0$;

(ii) *D has exactly one (simple) negative eigenvalue; and*

(iii) *the corresponding eigenvector is either semipositive or seminegative;*

(iv) $c \leqq 0$;

(v) *$c_i = 0$ if the* ith *row (column) of D is* 0.

We shall say a bit more about this result at the end of Section 5. However, for quadratic programming, an even more significant result is

THEOREM 3.7 [Martos, 8, Theorem 3]. *If the matrix*

$$\begin{bmatrix} D & c \\ c^T & 0 \end{bmatrix}$$

has no row of zeros and $\phi(x) = \frac{1}{2}x^TDx + c^Tx$ is quasi-convex, but non-convex, on the nonnegative orthant, it is pseudo-convex on the semipositive orthant.

Thus for a quadratic program of the form

$$\begin{aligned} &\text{minimize } \phi(x) = \tfrac{1}{2}x^TDx + c^Tx \\ &\text{subject to } Ax = b, \\ &\qquad\qquad\quad x \geqq 0 \end{aligned}$$

a nonzero Kuhn-Tucker stationary point x^* must be a global minimum of ϕ on the set $S = \{x | Ax = b, x \geqq 0\}$.

4. NEW CRITERIA FOR QUASI-CONVEXITY AND PSEUDO-CONVEXITY

Martos' results on quasi-convexity and pseudo-convexity of quadratic forms (over appropriate orthants) are related to positive subdefiniteness and strict positive subdefiniteness, respectively. Our object in this section is to establish necessary and sufficient conditions for a quadratic form $\psi(x) = x^TDx$ to be merely positive subdefinite. To this end, we give a converse for Theorem 3.1, a new set of equivalent conditions, and a new, simpler, set of sufficient conditions.

THEOREM 4.1. *The quadratic form $\psi(x) = x^TDx$ is merely positive subdefinite if and only if*

(i) $D \leqslant 0$,
(ii) *the spectrum of* D (spec D) *contains exactly one negative element.*

Proof. The necessity is just Theorem 3.1, so we prove only the sufficiency. Clearly D is not positive semidefinite, for otherwise $d_{ii} \geqq 0$ for all i and hence $d_{ii} = 0$ for all i. Hence $D \geqq 0$. But this contradicts $D \leqslant 0$. Therefore there exists a vector x such that $x^TDx < 0$. Suppose that Eq. (2.5) does not hold at x; i.e., Dx has components of opposite sign. Then there exists a vector $y > 0$ such that $y^TDx = 0$. Since D is real and symmetric, there exists an orthogonal matrix P such that $D = P\Delta P^T$, where Δ is a square matrix in which the diagonal entries are the elements of spec D and the off-diagonal entries are 0, and the columns of P are the corresponding eigenvectors. Since all the properties under consideration here are unaffected by principal rearrangement of D, we may assume that δ_1 is the unique negative element of spec D. Let $u = P^Tx$, $v = P^Ty$. Now we have

$$u^T\Delta u = x^TDx < 0,$$

$$v^T\Delta u = y^TDx = 0,$$

$$v^T\Delta v = y^TDy < 0.$$

But from the properties of Δ we can say

$$u^T\Delta u = \sum_{i=1}^{n} \delta_i u_i^2 < 0 \quad \text{implies} \quad u_1 \neq 0,$$

$$v^T\Delta v = \sum_{i=1}^{n} \delta_i v_i^2 < 0 \quad \text{implies} \quad v_1 \neq 0.$$

Let $\alpha = u_1/v_1$. Then

$$(u - \alpha v)^T\Delta(u - \alpha v) = u^T\Delta u - 2\alpha v^T\Delta u + \alpha^2 v^T\Delta v < 0.$$

On the other hand,

$$(u - \alpha v)^T\Delta(u - \alpha v) = \sum_{i=2}^{n} \delta_i(u_i - \alpha v_i)^2 \geqq 0.$$

This contradiction shows that Eq. (2.5) must hold.

Given the seminegativity property (i), one can replace the spectral property (ii) by an equivalent condition on the principal minors of D.

THEOREM 4.2. *If D is a real symmetric seminegative matrix, then the spectrum of D contains exactly one negative element if and only if D has nonpositive principal minors.*

Proof. Suppose $D \leqslant 0$ and D has nonpositive principal minors. We shall show that $\operatorname{spec} D$ contains exactly one negative element. Recall that D is of order n. Let

$$g(\lambda) = \det(D - \lambda I) = (-\lambda)^n + c_{n-1}(-\lambda)^{n-1} + \cdots + c_1(-\lambda) + c_0$$

be the characteristic polynomial of D. It is well known that the coefficient c_{n-k} of g is the sum of the k by k principal minors of D, $k = 0, 1, \ldots, n$. (See [4, p. 70].)

We digress momentarily to review *Descartes' Rule of Signs.* The number of positive real roots of a polynomial equation

$$g(\xi) = a_n\xi^n + a_{n-1}\xi^{n-1} + \cdots + a_1\xi + a_0 = 0$$

with real coefficients is never greater than the number of variations of its coefficients $a_0, a_1, \ldots, a_n$ (and, if less, differs by an even number). The number of variations is the number of sign changes (positive to negative or vice versa). See [12] for details.

The coefficients $c_0, c_1, \ldots, c_{n-1}$ are nonpositive, and at least one of them is negative (if not c_{n-1}, the trace of D, then c_{n-2}). Of course, $c_n = 1$ so the numbers of variations among $c_0, c_1, \ldots, c_n$ is precisely 1. By the Rule of Signs, $g(-\lambda) = 0$ has at most one positive root. Hence $g(\lambda) = 0$ has at most one negative root. But $g(\lambda) = 0$ has at least one negative root since D is not positive semidefinite. Thus $\operatorname{spec} D$ contains exactly one negative element.

For the converse, we assume $D \leqslant 0$; then $\operatorname{spec} D$ contains exactly one negative element. This, we know, implies that D is merely positive subdefinite. It also implies that $\det D \leqq 0$.

Clearly the principal submatrices of any positive subdefinite matrix are positive subdefinite (regardless of whether D is seminegative). While the inheritance of *mere* positive subdefiniteness is false, we can assert that, if D is merely positive subdefinite, its principal submatrices are nonpositive and positive subdefinite. If A is a principal submatrix of D, then either A is positive semidefinite, in which case $A = 0$ and $\det A = 0$ or A is merely positive subdefinite in which case $\det A \leqq 0$. This completes the proof.

Obviously, the prospect of computing all principal minors of a large matrix does not inspire much enthusiasm. On the other hand, for very small examples (orders 2 and 3), the test is almost by inspection.

In the light of obvious analogies with positive definiteness, and in the interest of a more efficient *sufficiency test,* we also prove

THEOREM 4.3. *If D is a real symmetric seminegative matrix with negative leading principal minors, then D is strictly merely positive subdefinite.*

Proof. Let $D_1, D_2, \ldots, D_n$ denote the leading principal minors of D. Thus

$$D_i = \det \begin{bmatrix} d_{11} & \cdots & d_{1i} \\ \vdots & & \vdots \\ d_{i1} & \cdots & d_{ii} \end{bmatrix} < 0, \qquad i = 1, \ldots, n.$$

Since D is real, symmetric, and nonsingular, it has n real, nonzero eigenvalues. Letting π and ν denote the number of positive and negative elements of spec D, respectively, we have $n = \pi + \nu$. By Jacobi's Theorem [4, p. 303], ν equals the number of variations in $\{1, D_1, \ldots, D_n\}$. Hence D has exactly one negative eigenvalue and $n - 1$ positive eigenvalues. Since D is seminegative, D is merely positive subdefinite and strictly so because D cannot contain a row of zeros.

As in the classical case of positive semidefiniteness, it is easily seen that one cannot vary the statement of Theorem 4.3 to obtain positive subdefiniteness by just relaxing the negativity of the leading principal minors to nonpositivity.

As we mentioned in Section 1, these criteria are related in spirit to the idea of giving finite sets of conditions for copositivity as in [3]. Indeed, there is an even deeper relationship between these classes of matrices, as indicated in the next theorem. To review briefly, a real (symmetric) matrix D is *copositive* if $x^TDx \geqq 0$ for every $x \geqq 0$. D is *copositive of order k* if every $k \times k$ principal submatrix of D is copositive. If D is copositive of order n, it is copositive of all lower orders, and copositivity of order 1 is just nonnegativity of the main diagonal elements. These facts make it possible to provide an inductive determinantal test for copositivity. Next we prove

THEOREM 4.4. *Let D be a real symmetric matrix of order n. D is copositive of order $n - 1$ but not copositive if and only if D^{-1} exists and is strictly merely positive subdefinite.*

Proof. By Theorem 2 in [3], the assumption that D is copositive of order $n - 1$ but not copositive implies (a) adj $D \geqq 0$, (b) $\det D < 0$. Hence D^{-1} exists and is seminegative. It clearly has no row of zeros. In the proof of the aforementioned theorem, it is shown that D (hence D^{-1}) has exactly one negative eigenvalue. Therefore D^{-1} is strictly merely positive subdefinite.

Conversely, if D^{-1} is strictly merely positive subdefinite, then

$$(-e)^T D^{-1}(-e) = e^T D^{-1} e < 0,$$

where $e^T = (1, \ldots, 1)$. We note that

$$D^{-1}(-e) > 0$$

and

$$-e^T D^{-1} D D^{-1}(-e) = e^T D e < 0;$$

hence D is not copositive. Now suppose some principal submatrix of D is not copositive. We may assume that it is the leading principal submatrix of order $n - 1$. Let $y^T = (y_1, \ldots, y_n) \geqslant 0$ with $y_n = 0$ satisfy $y^T D y < 0$. Then

$$y^T D y = y^T D D^{-1} D y < 0.$$

Since D^{-1} is strictly merely positive subdefinite, we must have $D^{-1} D y > 0$ or $D^{-1} D y < 0$; but neither of these is the case.

5. EXTENSION TO THE CASE OF QUADRATIC FUNCTIONS

It might be imagined that, once the quasi-convexity of the quadratic form ψ on E_+^n has been established, one has only to add on a linear function to obtain a quasi-convex quadratic function ϕ. But this is not the case. From Theorem 3.6 we know that, if $\phi(x) = \frac{1}{2} x^T D x + c^T x$ is quasi-convex but not convex on E_+^n, then $D \leqslant 0$, $c \leqq 0$, and $c_i = 0$ if the ith column of D is 0. Even under these restrictions one could hope to add on a linear term and preserve quasi-convexity on E_+^n. However, this is also inadequate. For example, the quadratic function

$$\phi(x_1, x_2) = \tfrac{1}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.1}$$

has a quadratic part which is quasi-convex on E_+^n and a linear part which satisfies the necessary conditions expressed in Theorem 3.6. Yet the quadratic function in (5.1) is not quasi-convex on E_+^n, as may be verified by noting that Eq. (2.1) fails to hold when

$$x = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Consequently, one must pay closer attention to the relationship between D and c.

The key to the test for quasi-convexity of a quadratic function $\phi(x) = \frac{1}{2}x^TDx + c^Tx$ on E_+^n lies partly in the often used observation that

$$\phi(x) \equiv \tfrac{1}{2}\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} D & c \\ c^T & 0 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix}. \tag{5.2}$$

Thus, if we set

$$\Psi(x, \xi) = \tfrac{1}{2}\begin{bmatrix} x \\ \xi \end{bmatrix}^T \begin{bmatrix} D & c \\ c^T & 0 \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad \begin{bmatrix} x \\ \xi \end{bmatrix} \in E_+^{n+1}, \tag{5.3}$$

then from (5.2) we have the relationship

$$\phi(x) = \Psi(x, 1). \tag{5.4}$$

This tends to motivate

THEOREM 5.1. *If the quadratic function $\phi(x) = \frac{1}{2}x^TDx + c^Tx$ is not convex on E^n, then ϕ is quasi-convex on E_+^n if and only if the quadratic form*

$$\Psi(x, \xi) = \tfrac{1}{2}\begin{pmatrix} x \\ \xi \end{pmatrix}^T \begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix}$$

is quasi-convex on E_+^{n+1} (or equivalently, merely positive subdefinite).

Proof. The sufficiency is immediate because of Eq. (5.4). To show the necessity, we prove, via Theorems 4.1 and 4.2, that Ψ is merely positive subdefinite. In particular, we show that

$$\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$$

must be seminegative and have nonpositive principal minors. The seminegativity has already been observed (see Theorem 3.6). It remains to check the signs of the principal minors. Those of D are nonpositive by Martos' theorem [8, Theorem 2] (Theorem 3.6 in our numbering) and Theorems 4.1 and 4.2. Moreover, a theorem of Arrow and Enthoven [1, Theorem 5] tells us, *mutatis mutandis*, that since our ϕ is quasi-convex on E_+^n, the leading principal minors of the matrix

$$\begin{bmatrix} 0 & x^TD + c^T \\ Dx + c & D \end{bmatrix}$$

are nonpositive for all x.

We may choose $x = 0$ and then claim that the leading principal minors of

$$\begin{bmatrix} 0 & c^T \\ c & D \end{bmatrix} \tag{5.5}$$

are nonpositive. After a suitable permutation of variables, any of the remaining principal minors in which we may be interested can be viewed as a leading principal minor in (5.5). Since this has no effect on the signs of the minors or the quasi-convexity, we obtain the result.

Naturally, we next wish to obtain a test for the pseudo-convexity of a quadratic function on the semipositive orthant. To this end, we remark that there is no loss of generality in assuming that

$$\phi(x) = \tfrac{1}{2}x^TDx + c^Tx, \qquad c \neq 0;$$

for, if $c = 0$, ϕ is just a quadratic *form* for which we already have a pseudo-convexity test.

Before proceeding, we shall need the following observation which is, in fact, the converse of Martos' theorem [8, Theorem 3] (Theorem 3.7 in our numbering).

LEMMA 5.1. *If θ is a continuous quasi-convex function on the nonempty convex set S in E^n, then θ is quasi-convex on $\bar{S}$, the closure of S.*

Proof. Let x and y be distinct points of $\bar{S}$. There exist sequences $\{x^i\}$ and $\{y^i\}$ in S converging to x and y, respectively. For each i, and for all $\lambda \in (0, 1)$,

$$\theta[\lambda x^i + (1 - \lambda)y^i] \leqq \max\{\theta(x^i), \theta(y^i)\}.$$

It now follows from the continuity that

$$\theta[\lambda x + (1 - \lambda)y] \leqq \max\{\theta(x), \theta(y)\},$$

and this implies the quasi-convexity of θ on $\bar{S}$.

It might seem that the pseudo-convexity of ϕ on $E_+^n\backslash 0$ implies that of

$$\Psi(x, \xi) = \frac{1}{2}\begin{pmatrix} x \\ \xi \end{pmatrix}^T \begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

on $E_+^{n+1}\backslash 0$. But this is not the case. To see this, we note that, if $D = (d_{ij})$ and $c^T = (c_1, \ldots, c_n)$ where

$$d_{ij} = \begin{cases} \delta < 0 & i = j = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad c_i = \begin{cases} \gamma < 0 & i = 1 \\ 0 & \text{otherwise} \end{cases},$$

then $\phi(x_1, \ldots, x_n) = \frac{1}{2}x^T Dx + c^T x = \frac{1}{2}\delta x_1^2 + \gamma x_1$ is pseudo-convex on $E_+^n\backslash 0$. However, if $n \geqslant 1$, $\Psi(x, \xi)$ is not pseudo-convex on E_+^{n+1} since

$$\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$$

contains at least one row of zeros.

THEOREM 5.2. *Let $\phi(x) = \frac{1}{2}x^T Dx + c^T x$ be a nonconvex quadratic function on E^n such that*

$$\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$$

contains no row of zeros. Then ϕ is pseudo-convex on $E_+^n\backslash 0$ if and only if

$$\Psi(x, \xi) = \frac{1}{2}\begin{pmatrix} x \\ \xi \end{pmatrix}^T \begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

is pseudo-convex on $E_+^{n+1}\backslash 0$ (or, equivalently, strictly merely positive subdefinite).

Proof. Suppose $\Psi(x, \xi)$ is pseudo-convex on $E_+^{n+1}\backslash 0$. Now let $x, y \in E_+^n\backslash 0$ satisfy

$$\nabla\phi(x)(y - x) = (x^T D + c^T)(y - x) \geqq 0.$$

Using the identity (5.4), we find that

$$\nabla\Psi(x, 1)\left[\begin{pmatrix} y \\ 1 \end{pmatrix} - \begin{pmatrix} x \\ 1 \end{pmatrix}\right] \geqq 0.$$

Hence, by the pseudo-convexity of Ψ,

$$\phi(y) = \Psi(y, 1) \geqq \Psi(x, 1) = \phi(x),$$

which shows that ϕ is pseudo-convex on $E_+{}^n \backslash 0$.

Conversely, let ϕ be pseudo-convex on $E_+{}^n \backslash 0$. By Lemma 5.1, ϕ is quasi-convex on $E_+{}^n$. Hence

$$\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$$

is merely positive subdefinite. By our assumption that this matrix has no row of zeros, it is strictly merely positive subdefinite, i.e., $\Psi(x, \xi)$ is pseudo-convex on $E_+{}^{n+1} \backslash 0$.

We close this section with a result that can be viewed as a generalization of Theorem 3.6. The proof follows the one given by Martos.

THEOREM 5.3. *Let the* $(n + 1) \times (n + 1)$ *matrix*

$$A = \begin{pmatrix} D & c \\ c^T & k \end{pmatrix}$$

be merely positive subdefinite. If D *is nonzero but has a row of zeros, then the corresponding row of* A *is zero.*

Proof. For simplicity, assume that the first row of D is zero. Let $e = (1, \ldots, 1)^T \in E^n$, $e_1 = (1, 0, \ldots, 0)^T \in E^n$, and $\beta \in E^1$. Define $v = e + \beta e_1$ and

$$\bar{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

We have

$$e_1{}^T D e_1 = 0, \qquad e^T D e < 0.$$

Hence, for all real β,

$$\bar{v}^T A \bar{v} = v^T D v = e^T D e < 0;$$

since A is merely positive subdefinite and $v > 0$, it follows that

$$A\bar{v} = \begin{bmatrix} Dv \\ c^T v \end{bmatrix} \leqslant 0.$$

In particular,

$$c_1 \beta + \sum_{i=1}^{n} c_i = c^T v \leqq 0 \quad \text{for all} \quad \beta \in E^1.$$

But this implies $c_1 = 0$.

By repeated application of this result we obtain

COROLLARY 5.1. *If any row in a nonzero principal submatrix of a merely positive subdefinite matrix equals zero, then the corresponding row of the entire matrix equals zero.*

6. NONCONVEX QUADRATIC PROGRAMMING

The notions of quasi-convexity and pseudo-convexity have arisen in the context of mathematical economics and mathematical programming. In a sense, they represent limiting conditions under which some desirable properties can validly be asserted. The work of Martos [8] has shown that nonconvex quadratic functions which happen to be quasi-convex on the nonnegative orthant are actually pseudo-convex on the semipositive orthant.

One implication of this interesting result is that, for such an objective function, any quadratic programming method which calculates a Kuhn-Tucker stationary point produces a globally optimal solution to a non-convex problem. The authors hope that the characterizations provided here may one day find usage in the testing of quadratic functions as a prelude to the application of Martos' theorem.

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